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Gauge equivalence between $(2+1)$-dimensional continuous Heisenberg ferromagnetic models and nonlinear Schrödinger-type equations

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Abstract. The gauge equivalence between the $(2+1)$-dimensional Zakharov equation and the $(2+1)$-dimensional integrable continuous Heisenberg ferromagnetic model is established. Their integrable reductions are also shown explicitly.

1. Introduction

The concepts of gauge equivalence between completely integrable partial differential equations (NPDE) play an important role in the theory of solitons [1, 2]. So, for example, in $(1+1)$-dimensional soliton theory, a well known gauge equivalence takes place between the Landau–Lifshitz equation (LLE) or the $(1+1)$-dimensional continuous Heisenberg ferromagnetic model

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \boldsymbol{S}_{xx} \tag{1}$$

and the nonlinear Schrödinger equation (NLSE)

$$\mathrm{i}q_t + q_{xx} + 2E|q|^2 q = 0 \tag{2}$$

where $E = \pm 1$ [1–3].

Many efforts have recently been made to study the $(2+1)$-dimensional integrable NPDE [4–11]. Here we have the following interesting phenomenon: for every $(1+1)$-dimensional soliton (integrable) equation, there exist several $(2+1)$-dimensional integrable (and nonintegrable) generalizations. So, for example, the LLE (1) admits the following $(2+1)$-dimensional integrable and nonintegrable extensions. (Below we use the conditional notations, e.g. the M-I equation or the M-IX equation, etc, in order to distinguish the different spin systems.)

(1°) The M-I equation [6]

$$\boldsymbol{S}_t = (\boldsymbol{S} \wedge \boldsymbol{S}_y + u\boldsymbol{S})_x \tag{3a}$$

$$u_x = -\boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{3b}$$

(2°) The M-VIII equation [6]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \boldsymbol{S}_{xx} + u\boldsymbol{S}_x \tag{4a}$$

$$u_y = \kappa \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{4b}$$

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(3°) The Ishimori equation [9]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge (\boldsymbol{S}_{xx} + \alpha^2 \boldsymbol{S}_{yy}) + u_y \boldsymbol{S}_x + u_x \boldsymbol{S}_y \tag{5a}$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{5b}$$

(4°) The M-IX equation [6]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge M_1 \boldsymbol{S} + A_2 \boldsymbol{S}_x + A_1 \boldsymbol{S}_y \tag{6a}$$

$$M_2 u = -2\alpha^2 \kappa \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{6b}$$

(5°) The M-XVIII equation [6]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \{\boldsymbol{S}_{xx} - \alpha(2b+1)\boldsymbol{S}_{xy} + \alpha^2 \boldsymbol{S}_{yy}\} + A'_{20} \boldsymbol{S}_x + A'_{10} \boldsymbol{S}_y \tag{7a}$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \kappa \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{7b}$$

(6°) The M-XIX equation [6]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \{\alpha^2 \boldsymbol{S}_{yy} - a(a+1)\boldsymbol{S}_{xx}\} + A''_{20} \boldsymbol{S}_x + A''_{10} \boldsymbol{S}_y \tag{8a}$$

$$M_2 u = -2\alpha^2 \kappa \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{8b}$$

(7°) The M-XX equation [6]

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \{(b+1)\boldsymbol{S}_{xx} - b\boldsymbol{S}_{yy}\} + (b+1)u_x \boldsymbol{S}_x + b u_y \boldsymbol{S}_y \tag{9a}$$

$$u_{xy} = \alpha\kappa \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{9b}$$

(8°) The $(2+1)$-dimensional LLE

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge (\boldsymbol{S}_{xx} + \boldsymbol{S}_{yy}) \tag{10}$$

and so on. Here $\boldsymbol{S} = (S_1, S_2, S_3)$, $\boldsymbol{S}^2 = 1$, a, b, α, κ are constants, u is a scalar function. All of these equations in $(1+1)$-dimensions reduce to the LLE (1). Note that here equations (3)–(9) are integrable, at the same time equation (10) is apparently not integrable. Besides which, all of these spin systems have a remarkable common property, namely, they possess the topological invariant

$$Q = \frac{1}{4\pi} \int\int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{d}y\, \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y). \tag{11}$$

The solutions of these spin systems are therefore classified by the integer value of $Q = N = 0, \pm 1, \pm 2, \pm 3, \ldots$.

In $2+1$ dimensions, gauge equivalence has been recently constructed for the Davey–Stewartson and Ishimori equations [5], for other spin systems, nonlinear Schrödinger-type equations etc [6–8]. Here, in particular, the following two questions naturally arise.

(1′) What equations are gauge equivalent counterparts of the equations (3), (4) and (6)–(9)?

(2′) What equations are the $(2+1)$-dimensional integrable generalizations of the anisotropic LLE?

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge \boldsymbol{S}_{xx} + \boldsymbol{S} \wedge J\boldsymbol{S} \tag{12}$$

where $J = \mathrm{diag}(J_1, J_2, J_3)$ is the matrix of anisotropy. In this work we try to provide answers to these guestions.

This paper is organized as follows. In section 2 we establish the gauge equivalence between the M-IX equation and the Zakharov equation. In section 3 we construct the integrable reductions of the M-IX equation and present their equivalent counterparts. In section 4 we consider the $(2+1)$-dimensional continuous Heisenberg ferromagnet model with the one-ion anisotropy and obtain its equivalent soliton equation. Also, we establish the gauge equivalence between the isotropic and anisotropic versions of this model. We finish with a conclusion.

2. Gauge equivalence between the M-IX equation and the Zakharov equation

The M-IX equation looks like [6]

$$\mathrm{i}S_t + \tfrac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0 \tag{13a}$$

$$M_2 u = \frac{\alpha^2}{4\mathrm{i}}\,\mathrm{tr}(S[S_y, S_x]) \tag{13b}$$

where $\alpha, b, a =$ constants and

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix} \qquad S^{\pm} = S_1 \pm \mathrm{i}S_2 \qquad S^2 = I \qquad r^2 = \pm 1$$

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 2\alpha(b-a)\frac{\partial^2}{\partial x \partial y} + (a^2 - 2ab - b)\frac{\partial^2}{\partial x^2}$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a+1)\frac{\partial^2}{\partial x \partial y} + a(a+1)\frac{\partial^2}{\partial x^2}$$

$$A_1 = 2\mathrm{i}\{\alpha(2b+1)u_y - (2ab+a+b)u_x\}$$

$$A_2 = 2\mathrm{i}\{\alpha^{-1}(2a^2 b + a^2 + 2ab + b)u_x - (2ab+a+b)u_y\}.$$

This set of equations is integrable in the sense that it admits the Lax representation and has different types of solutions. In general, we will distinguish the two integrable cases: the M-IXA equation as $\alpha^2 = 1$ and the M-IXB equation as $\alpha^2 = -1$. Besides, the equation (13) admits several integrable reductions: the M-VIII equation as $b = 0$, the Ishimori equation as $a = b = -\frac{1}{2}$ and so on. Equation (13) is the $(2+1)$-dimensional integrable generalization of the LLE (1) and in $1+1$ dimensions reduces to it.

The Lax representation of the equation (13) is given by [6]

$$\alpha\Phi_y = \tfrac{1}{2}[S + (2a+1)I]\Phi_x \tag{14a}$$

$$\Phi_t = \frac{\mathrm{i}}{2}[S + (2b+1)I]\Phi_{xx} + \frac{\mathrm{i}}{2}W\Phi_x \tag{14b}$$

with

$$W = (2b+1)E + (2b - a + \tfrac{1}{2})SS_x + (2b+1)FS + FI + \tfrac{1}{2}S_x + ES + \alpha SS_y$$

$$E = -\frac{\mathrm{i}}{2\alpha}u_x \qquad F = \frac{\mathrm{i}}{2}\left(\frac{(2a+1)u_x}{\alpha} - 2u_y\right).$$

Let us now find the equation which is gauge equivalent to equation (13). To this end, we consider the following transformation

$$\Phi = g^{-1}\Psi \tag{15}$$

where Φ is the matrix solution of linear problem (14), Ψ and g are temporally unknown matrix-valued functions. Substituting (15) into (14) we get

$$\alpha\Psi_y = \tfrac{1}{2}[gSg^{-1} + (2a+1)I]\Psi_x + [\alpha g_y - \tfrac{1}{2}gSg^{-1}g_x - \tfrac{1}{2}(2a+1)g_x]g^{-1}\Psi \tag{16a}$$

$$\begin{aligned}\Psi_t = \frac{\mathrm{i}}{2}[gSg^{-1} + (2b+1)I]\Psi_{xx} &+ g\left\{\mathrm{i}[S + (2b+1)I](g^{-1})_x + \frac{\mathrm{i}}{2}Wg^{-1}\right\}\Psi_x \\ &+ g\left\{g_t g^{-1} + \frac{\mathrm{i}}{2}[S + (2b+1)I](g^{-1}_{xx}) + \frac{\mathrm{i}}{2}W(g^{-1})_x\right\}\Psi.\end{aligned} \tag{16b}$$

Now let us choose the unknown functions g and S in the form

$$g = \begin{pmatrix} f_1(1+S_3) & f_1 rS^- \\ f_2 rS^+ & -f_2(1+S_3) \end{pmatrix} \qquad S = g^{-1}\sigma_3 g \tag{17}$$

where f_j satisfy the following equations:

$$\alpha(\ln f_1)_y - (a+1)(\ln f_1)_x = \frac{(a+1)(S_{3x} + S_3 S_{3x} + S_x^- S^+) - \alpha(S_{3y} + S_3 S_{3y} + S_y^- S^+)}{2(1+S_3)} \tag{18a}$$

$$\alpha(\ln f_2)_y - a(\ln f_2)_x = \frac{a(S_{3x} + S_3 S_{3x} + S_x^+ S^-) - \alpha(S_{3y} + S_3 S_{3y} + S_y^+ S^-)}{2(1+S_3)}. \tag{18b}$$

From (16), (17) it follows that

$$\alpha g_y g^{-1} - B_1 g_x g^{-1} = B_0 \tag{19}$$

where

$$B_1 = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix} \qquad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$

Here $a =$ constant and p, q are some complex functions which are equal to

$$q = \frac{f_1\{\alpha[S_{3y} S^- - S_y^-(1+S_3)] + (a+1)[S_{3x} S^- - S_x^-(1+S_3)]\}}{2 f_2 (1+S_3)} \tag{20a}$$

$$p = \frac{f_2\{a[S_{3x} S^+ - S_x^-(1+S_3)] + \alpha[S_{3y} S^+ - S_y^-(1+S_3)]\}}{2 f_1 (1+S_3)}. \tag{20b}$$

Hence we obtain

$$pq = \tfrac{1}{4}\{\alpha(2a+1)\boldsymbol{S}_x \boldsymbol{S}_y + \mathrm{i}\alpha \boldsymbol{S}(\boldsymbol{S}_x \wedge \boldsymbol{S}_y) - a(a+1)\boldsymbol{S}_x^2 - \alpha^2 \boldsymbol{S}_y^2\} \tag{21}$$

where $\boldsymbol{S} = (S_1, S_2, S_3)$ is the spin vector, $\boldsymbol{S}^2 = 1$. After these calculations equations (16) take the forms

$$\alpha \Psi_y = B_1 \Psi_x + B_0 \Psi \tag{22a}$$

$$\Psi_t = \mathrm{i} C_2 \Psi_{xx} + C_1 \Psi_x + C_0 \Psi \tag{22b}$$

with

$$C_2 = \begin{pmatrix} b+1 & 0 \\ 0 & b \end{pmatrix} \qquad C_1 = \begin{pmatrix} 0 & \mathrm{i}q \\ \mathrm{i}p & 0 \end{pmatrix} \qquad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$c_{12} = \mathrm{i}(2b - a + 1) q_x + \mathrm{i}\alpha q_y \qquad c_{21} = \mathrm{i}(a - 2b) p_x - \mathrm{i}\alpha p_y.$$

Here c_{jj} is the solution of the following equations

$$\begin{aligned} (a+1)c_{11x} - \alpha c_{11y} &= \mathrm{i}[(2b - a + 1)(pq)_x + \alpha(pq)_y] \\ a c_{22x} - \alpha c_{22y} &= \mathrm{i}[(a - 2b)(pq)_x - \alpha(pq)_y]. \end{aligned} \tag{23}$$

The compatibility condition of equations (22) gives the following $(2+1)$-dimensional nonlinear Schrödinger-type equation

$$\mathrm{i} q_t + M_1 q + v q = 0 \tag{24a}$$

$$\mathrm{i} p_t - M_1 p - v p = 0 \tag{24b}$$

$$M_2 v = -2 M_1 (pq) \tag{24c}$$

which is the Zakharov equation (ZE) [10], where $v = \mathrm{i}(c_{11} - c_{22})$ and $p = r^2 q$. Thus we have proved that equation (13) is gauge equivalent to the $(2+1)$-dimensional ZE (24) and vice versa. Note that the gauge transformation presented above is reversible. In fact, starting from the linear problem (22) after the standard gauge transformation, we can come to the Lax representation (14) of equation (13) that proves the gauge equivalence between equations (13) and (24).

3. Integrable reductions

It is interesting to note that equation (13) admits some integrable reductions. Let us now consider these particular integrable cases.

3.1. The M-VIII equation

Let $b = 0$. Then equations (13) take the form

$$\mathrm{i}S_t = \tfrac{1}{2}[S_{\xi\xi}, S] + \mathrm{i}wS_\xi \tag{25a}$$

$$w_\eta = \frac{1}{4\mathrm{i}}\,\mathrm{tr}(S[S_\eta, S_\xi]) \tag{25b}$$

where

$$\xi = x + \frac{a+1}{\alpha}y \qquad \eta = -x - \frac{a}{\alpha}y \qquad w = -\frac{1}{\alpha}u_\xi$$

which is the M-VIII equation [6]. The corresponding Lax representation has the form

$$\Phi_{Z^+} = S\Phi_{Z^-} \tag{26a}$$

$$\Phi_t = 2\mathrm{i}[S+I]\Phi_{Z^-Z^-} + \mathrm{i}K\Phi_{Z^-} \tag{26b}$$

where $Z^\pm = \xi \pm \eta$ and

$$K = E_0 + E_0S + S_{Z^-} + 2SS_{Z^-} + SS_{Z^+} \qquad E_0 = \frac{\mathrm{i}}{\alpha}(u_{Z^+} + u_{Z^-}).$$

The gauge equivalent counterpart of equation (25), we obtain from (24) as $b = 0$

$$\mathrm{i}q_t + q_{\xi\xi} + vq = 0 \tag{27a}$$

$$v_\eta = -2r^2(\bar{q}q)_\xi \tag{27b}$$

which is the other ZE [10].

3.2. The Ishimori equation

Now let us consider the case: $a = b = -\frac{1}{2}$. In this case equations (13) reduce to the well known Ishimori equation

$$\mathrm{i}S_t + \tfrac{1}{2}[S, (\tfrac{1}{4}S_{xx} + \alpha^2 S_{yy})] + \mathrm{i}u_yS_x + \mathrm{i}u_xS_y = 0 \tag{28a}$$

$$\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4\mathrm{i}}\,\mathrm{tr}(S[S_y, S_x]). \tag{28b}$$

The Ishimori equation (28) is of great interest, since it is the first example of integrable spin systems on the plane. This equation is considered as a useful 'laboratory' for experiments with new theoretical tools to reveal the specific nature of soliton models of $(2+1)$-dimensional spin systems. As is well known, equation (28) allows us to obtain the rich class of topologically nontrivial and nonequivalent solutions (solitons, lumps, vortex, dromions and so on) which are classified by the value of the topological charge (11).

The gauge equivalent counterpart of equation (28) is the Davey–Stewartson equation

$$\mathrm{i}q_t + \tfrac{1}{4}q_{xx} + \alpha^2 q_{yy} + vq = 0 \tag{29a}$$

$$\alpha^2 v_{yy} - \tfrac{1}{4}v_{xx} = -2\{\alpha^2(pq)_{yy} + \tfrac{1}{4}(pq)_{xx}\} \tag{29b}$$

that follows from the ZE (24). This fact was first established in [5]. The Lax representations of (28) and (29) can be obtained from (14) and (22), respectively, as $a = b = -\frac{1}{2}$.

3.3. The M-XVIII equation

Now we consider the reduction: $a = -\frac{1}{2}$. Equation (13) then reduces to the M-XVIII equation [6]

$$\mathrm{i}S_t + \tfrac{1}{2}[S, (\tfrac{1}{4}S_{xx} - \alpha(2b+1)S_{xy} + \alpha^2 S_{yy})] + A_2' S_x + A_1' S_y = 0 \tag{30a}$$

$$\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4\mathrm{i}}\,\mathrm{tr}(S[S_y, S_x]) \tag{30b}$$

where $A_j' = A_j$ as $a = -\frac{1}{2}$. The corresponding gauge equivalent equation is obtained from (24) and looks like

$$\mathrm{i}q_t + \tfrac{1}{4}q_{xx} - \alpha(2b+1)q_{xy} + \alpha^2 q_{yy} + vq = 0 \tag{31a}$$

$$\alpha^2 v_{yy} - \tfrac{1}{4}v_{xx} = -2\{\alpha^2(pq)_{yy} - \alpha(2b+1)(pq)_{xy} + \tfrac{1}{4}(pq)_{xx}\}. \tag{31b}$$

From (14) and (22) we obtain the Lax representations of (30) and (31), respectively, as $a = -\frac{1}{2}$.

3.4. The M-XIX equation

Let us consider the case: $a = b$. Then we obtain the M-XIX equation [6]

$$\mathrm{i}S_t = \tfrac{1}{2}[S, \{\alpha^2 S_{yy} - a(a+1)S_{xx}\}] + A_2'' S_x + A_1'' S_y \tag{32a}$$

$$M_2 u = -\frac{\alpha^2}{4\mathrm{i}}\,\mathrm{tr}(S[S_x, S_y]) \tag{32b}$$

where $A_j'' = A_j$ as $a = b$. The corressponding NLSE has the form

$$\mathrm{i}q_t + \alpha^2 q_{yy} - a(a+1)q_{xx} + vq = 0 \tag{33a}$$

$$M_2 v = -2[\alpha^2(|q|^2)_{yy} - a(a+1)(|q|^2)_{xx}]. \tag{33b}$$

The Lax representations of these equations are obtained from (14) and (22), respectively, as $a = b$.

3.5. The M-XX equation

This equation is read as [6]

$$\mathrm{i}S_t = \tfrac{1}{2}[S, bS_{\eta\eta} - (b+1)S_{\xi\xi}] + w_\eta S_\eta + w_\xi S_\xi \tag{34a}$$

$$w_{\xi\eta} = -\frac{1}{4\mathrm{i}}\,\mathrm{tr}(S[S_\eta, S_\xi]) \tag{34b}$$

where $w = -\alpha^{-1}u$. The associated linear problem is given by

$$\Phi_{Z^+} = S\Phi_{Z^-} \tag{35a}$$

$$\Phi_t = 2\mathrm{i}[S + (2b+1)I]\Phi_{Z^-Z^-} + \mathrm{i}W_0\Phi_{Z^-} \tag{35b}$$

where $Z^\pm = \xi \pm \eta$ and

$$W_0 = (2b+1)(E + FS) + F + ES + \tfrac{1}{2}S_{Z^-} + 2(2b+1)SS_{Z^-} + SS_{Z^+}$$

$$E = \frac{\mathrm{i}}{\alpha}u_{Z^-} \qquad F = \frac{\mathrm{i}}{\alpha}u_{Z^+}.$$

The gauge equivalent equation looks like

$$\mathrm{i}q_t + (1+b)q_{\xi\xi} - bq_{\eta\eta} + vq = 0 \tag{36a}$$

$$v_{\xi\eta} = -2\{(1+b)(pq)_{\xi\xi} - b(pq)_{\eta\eta}\}. \tag{36b}$$

This equation is integrated by the linear problem

$$f_{Z^+} = \sigma_3 f_{Z^-} + B_0 f \tag{37a}$$

$$f_t = 4\mathrm{i}C_2 f_{Z^-Z^-} + 2C_1 f_{Z^-} + C_0 f. \tag{37b}$$

where B_0, C_j are given as in (19).

Thus, we have presented some of the reductions of equation (13). All of these reductions are integrable in the sense that they admit the Lax representations.

4. The $(2+1)$-dimensional integrable spin system with anisotropy

4.1. *Gauge equivalent counterpart of the anisotropic spin system*

As an integrable system, the anisotropic LLE (12) can admit several integrable $(2+1)$-dimensional extensions [6]. One such integrable $(2+1)$-dimensional extension of the LLE (12) as $J_1 = J_2 = 0$, $J_3 = \triangle$ is the following M-I equation with one-ion anisotropy

$$\boldsymbol{S}_t = (\boldsymbol{S} \wedge \boldsymbol{S}_y + u\boldsymbol{S})_x + v\boldsymbol{S} \wedge \boldsymbol{n} \tag{38a}$$

$$u_x = -\boldsymbol{S} \cdot (\boldsymbol{S}_x \wedge \boldsymbol{S}_y) \tag{38b}$$

$$v_x = \triangle(\boldsymbol{S}_y \cdot \boldsymbol{n}) \tag{38c}$$

where u and v are scalar functions, $\boldsymbol{n} = (0, 0, 1)$, and $\triangle < 0$ and $\triangle > 0$ correspond respectively to the system with an easy plane and to that with an easy axis. Note that if the symmetry $\partial_x = \partial_y$ is imposed then the M-I equation (38) reduces to the well known LLE with single-site anisotropy

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge (\boldsymbol{S}_{xx} + \triangle(\boldsymbol{S} \cdot \boldsymbol{n})\boldsymbol{n}) \tag{39}$$

which is the particular case of the LLE (12) as $J_1 = J_2 = 0, L_3 = \triangle$. On the other hand, in the case when $\triangle = 0$, equation (38) becomes the isotropic M-I equation (3). It is known that equation (39) is gauge equivalent to the NLSE (2) [12–15]. In this section we construct the NLSE which is gauge equivalent to equation (38) with easy-axis anisotropy ($\triangle > 0$).

The Lax representation of the equation (38) may be given by [6]

$$\psi_x = L_1\psi \qquad \psi_t = 2\lambda\psi_y + M_1\psi \tag{40}$$

where

$$L_1 = \mathrm{i}\lambda S + \mu[\sigma_3, S] \qquad M_1 = 2\lambda A + 2\mathrm{i}\mu[A, \sigma_3] + 4\mathrm{i}\mu^2\{\sigma_3, V\}\sigma_3 \tag{41}$$

with

$$S = \sum_{k=1}^{3} S_k\sigma_k \qquad A = \tfrac{1}{4}([S, S_y] + 2\mathrm{i}uS) \qquad \mu = \sqrt{\frac{\triangle}{4}} \qquad \triangle > 0$$

and

$$V = \triangle \int_{-\infty}^{x} S_y \, \mathrm{d}x. \tag{42}$$

Here σ_k is a Pauli matrix, [,] ({,}) denoting the commutator (anticommutator), and λ is a spectral parameter. The matrix S has the following properties: $S^2 = I$, $S^* = S$, $\operatorname{tr} S = 0$. The compatibility condition of system (40) $\psi_{xt} = \psi_{tx}$ gives equation (38). Let us now consider the gauge transformation induced by $g(x, y, t) : \psi = g^{-1}\phi$, where $g^* = g^{-1} \in SU(2)$. It follows from the properties of the matrix S that it can be represented in the form $S = g^{-1}\sigma_3 g$. The new gauge equivalent operators L_2, M_2 are given by

$$L_2 = gL_1g^{-1} + g_xg^{-1} \qquad M_2 = gM_1g^{-1} + g_tg^{-1} - 2\lambda g_yg^{-1} \tag{43}$$

and satisfy the following system of equations

$$\phi_x = L_2\phi \qquad \phi_t = 2\lambda\phi_y + M_2\phi. \tag{44}$$

Now choosing

$$g_x g^{-1} + \mu g[\sigma_3, S]g^{-1} = U_0 \qquad gSg^{-1} = \sigma_3 \tag{45a}$$

$$g_t g^{-1} + 2\mathrm{i}\mu g[A, \sigma_3]g^{-1} + 4\mathrm{i}\mu^2 g\{\sigma_3, V\}\sigma_3 g^{-1} = V_0 \tag{45b}$$

with

$$U_0 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \qquad V_0 = \mathrm{i}\sigma_3(\partial_x^{-1}|q|_y^2 - U_{0y})$$

where $q(x, y, t)$ are the new complex-valued fields. Hence we finally obtain

$$L_2 = \mathrm{i}\lambda\sigma_3 + U_0 \qquad M_2 = V_0. \tag{46}$$

The compatibility condition $\phi_{xt} = \phi_{tx}$ of system (44) with operators L_2, M_2 (46) leads to the $(2+1)$-dimensional NLSE [10, 11]

$$\mathrm{i}q_t = q_{xy} + wq \qquad w_x = 2(|q|^2)_y. \tag{47}$$

This equation, under the reduction $\partial_y = \partial_x$, becomes the well known (1+1)-dimensional NLSE (2) as $E = +1$. Thus we have shown that the M-I equation with single-site anisotropy (38) is gauge equivalent to the $(2+1)$-dimensional NLSE—the ZE (47).

4.2. The isotropic and anisotropic spin systems: gauge equivalence

It is already known that equations (47) are gauge and geometrically equivalent to the isotropic M-I equation [7, 16–18]

$$\mathrm{i}S'_t = \tfrac{1}{2}([S', S'_y] + 2\mathrm{i}u'S')_x \tag{48a}$$

$$u'_x = -\frac{1}{4\mathrm{i}}\,\mathrm{tr}(S'[S'_x, S'_y]) \tag{48b}$$

which was introduced in [6] and arises from the compatibility condition of the linear problem

$$f_x = L'_1 f \qquad f_t = 2\lambda f_y + \lambda M'_1 f \tag{49}$$

where

$$L'_1 = \mathrm{i}\lambda S' \qquad M'_1 = \tfrac{1}{2}([S', S'_y] + 2\mathrm{i}u'S'). \tag{50}$$

Now we show that between the isotropic (48) and anisotropic (38) versions of the M-I equation the gauge equivalence takes place. Indeed the Lax representations, (40) and (49), which reproduce equations (38) and (48), respectively, can be obtained from each other by the λ-independent gauge transformation

$$L'_1 = hL_1h^{-1} + h_xh^{-1} \qquad M'_1 = hM_1h^{-1} + h_th^{-1} \tag{51}$$

with $h(x, y, t) = \psi^{-1}|_{\lambda=\lambda_0}$, where λ_0 is some fixed value of the spectral parameter λ.

So, the solutions of equations (38) and (48) are connected with each other by formulae $S = h^{-1}S'h$. Now we present the important relations between the field variables q and S:

$$|q|^2 = \tfrac{1}{2}[\boldsymbol{S}_x^2 - 8\mu S_{3x} + 16\mu^2(1 - S_2^3)] \tag{52a}$$

$$\bar{q}_x q - \bar{q}q_x = \frac{\mathrm{i}}{4}\boldsymbol{S}\cdot[\boldsymbol{S}_{xx} + 16\mu^2(\boldsymbol{S}\cdot\boldsymbol{n})\boldsymbol{n})] + 4\mu\boldsymbol{S}\cdot(\boldsymbol{S}_{xx}\wedge\boldsymbol{n}). \tag{52b}$$

These relations coincide with the corresponding connections between q and S from the one-dimensional case [12, 13].

Note that equation (38) with ($\triangle < 0$) easy plane single-site anisotropy is gauge equivalent to the following general $(2+1)$-dimensional NLSE [10, 11]

$$\mathrm{i}q_t = q_{xy} + wq \tag{53a}$$

$$\mathrm{i}p_t = -p_{xy} - wp \tag{53b}$$

$$w_x = 2(pq)_y. \tag{53c}$$

Besides, it can be shown that equation (38), when $S \in SU(1,1)/U(1)$, i.e. the non-compact group case, is gauge equivalent to the NLSE (53) with the repulsive interaction, $p = -\bar{q}$.

Finally, we note that the M-I equation (38) is the particular case of the M-III equation

$$\boldsymbol{S}_t = (\boldsymbol{S} \wedge \boldsymbol{S}_y + u\boldsymbol{S})_x + 2b(cb+d)\boldsymbol{S}_y - 4cv\boldsymbol{S}_x + \boldsymbol{S} \wedge \boldsymbol{V} \tag{54a}$$

$$u_x = -\boldsymbol{S}\dot{(}\boldsymbol{S}_x \wedge \boldsymbol{S}_y) \tag{54b}$$

$$v_x = \frac{1}{4(2bc+d)^2}(\boldsymbol{S}_x^2)_y \tag{54c}$$

$$\boldsymbol{V}_x = J\boldsymbol{S}_y. \tag{54d}$$

Note that these equations admit some integrable reductions: (a) the isotropic M-I equation as $c = J = 0$; (b) the anisotropic M-I equation as $c = J_1 = J_2 = 0$; (c) the M-II equation as $d = J_j = 0$ and; (d) the isotropic M-III equation as $J = 0$ and so on [6].

5. Concluding remarks

We have established the gauge equivalence between the $(2+1)$-dimensional classical continuous Heisenberg spin chain, the so-called M-IX equation (13) and the $(2+1)$-dimensional Zakharov equation (24). We have also presented their integrable reductions.

Another interesting concept in the theory of NLDE, and in particular spin systems, is the so-called the Lakshmanan equivalence [3]. The point is that, as shown by Lakshmanan [3], equations (1) and (2) are equivalent to each other in the geometrical sense. This equivalence, between spin systems and the corresponding NLSE, we call the Lakshmanan equivalence or L-equivalence. The L-equivalence between some $(2+1)$-dimensional spin systems and the corresponding NLSE, in particular between equations (3) and (47), is constructed in [6, 8, 16–18]. This problem deserves an individual and more detailed examination and we will discuss it elsewhere (see, e.g. [6, 8]).

Finally, regarding equations (38), (47) and (48) we should note the following fact. A spectral parameter λ, in contrast with the (1+1)-dimensional case where $\lambda_t = 0$, in our case satisfies the following nonlinear equation

$$\lambda_t = 2\lambda\lambda_y. \tag{55}$$

We can solve this equation using the following Lax representation

$$h_x = \mathrm{i}\lambda\sigma_3 h \qquad h_t = 2\lambda h_y. \tag{56}$$

The trivial solution is $\lambda = \lambda_1 =$ constant. Let us find the other non-trivial solutions. We consider the following general equation

$$\lambda_t = \kappa\lambda^n\lambda_y \tag{57}$$

where $\kappa =$ constant. Let us assume that

$$\lambda_y = \sum_j b_j\lambda^j \qquad \lambda_t = \sum_j d_j\lambda^j \tag{58}$$

where b_j, d_j are some functions in general of y, t. In particular, we can take

$$\lambda_t = \frac{\lambda}{a - \kappa t} \qquad \lambda_y = \frac{\lambda}{y + c} \tag{59}$$

where $a(c)$ is real (complex) constant. Hence, it follows that the solution of equation (57) has the form

$$\lambda = \lambda_2 = \left(\frac{y + c}{a - \kappa t}\right)^{\frac{1}{n}}. \tag{60}$$

So, if $n = 1$, we have

$$\lambda_2 = \frac{y + c}{a - \kappa t}. \tag{61}$$

If $n = 2$, we have

$$\lambda_2 = \left(\frac{y + c}{a - \kappa t}\right)^{\frac{1}{2}} \tag{62}$$

and so on. In our case the solution of (55) has the form (61) with $\kappa = 2$. It is necessary to note that unlike the $1 + 1$ dimensions, where $\lambda_t = 0$, in $2 + 1$ dimensions, we have the following integral of motion for the spectral parameter

$$K = \int \lambda \, \mathrm{d}y \qquad K_t = 0. \tag{63}$$

Finally, we note the corresponding solutions of the soliton equations are called the overlapping or breaking solutions [19]. In this case soliton equations must be solved with the help of the non-isospectral version of the inverse scattering transform (IST) method.

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